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WARPED SURFACES

Notes on Warped Surfaces.

Prepared for the use of Students
in
Civil Engineering
in the
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General Classification of Surfaces

(1). Surfaces capable of generation by right lines, and therefor having rectilinear elements:

(a) **Planes**, which may be generated by a right line moving upon some other right line, all positions of the generatrix being parallel to one another.

(b) **Single-Curved Surfaces**, which may be generated by a right line moving upon a curved line so that any two consecutive positions are either parallel, as in the cylinder, or intersect, as in the cone.

In the cone, all the positions intersect in a common point; but a single-curved surface may be generated, in which only consecutive positions of the generatrix shall intersect, no three positions meeting in a common point. A system of tangents to a line of double curvature will give such a surface.

(c) **Warped Surfaces**, which may be generated by a right line moving so that no two consecutive positions are parallel or intersect; in other words, no two consecutive positions of the generatrix are in the same plane.

(2). Surfaces capable of being generated by curves only, and therefor having no rectilinear elements. These are known as **Double-Curved Surfaces**.

Tangent Lines and Tangent Planes.

A right line is to be considered tangent to a curve when it contains two points of the curve infinitely near each other. For convenience, such position of points or of generating lines will be termed consecutive.

A tangent to a curve may thus be regarded as a limiting position of a secant, revolved about one of the points in which it cuts the curve.

For the purposes of demonstration it is sometimes convenient to consider a right line as a tangent to itself.

A plane may be defined as tangent to a curved surface at a point, when it contains the tangents to all the curves that can be drawn upon the curved surface through the point in question.

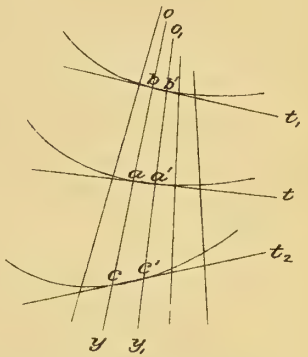
Developable Surfaces.

A surface is developable that can be generated by a right line so moving that any two of its consecutive positions are in the same plane.

From this definition, which may be confirmed by rigid proof, it is clear that warped and double-curved surfaces are not developable, except approximately; in other words, they cannot be unrolled upon a tangent plane without breaking the continuity of their surface.

Theorem: A plane tangent to a developable surface at any point, is tangent all along the element passing through that point.

Let oy and $o_1 y_1$ be consecutive positions of the right line generatrix of a given curved surface. At a let a plane be tangent to the given surface: then will it be tangent all along oy.

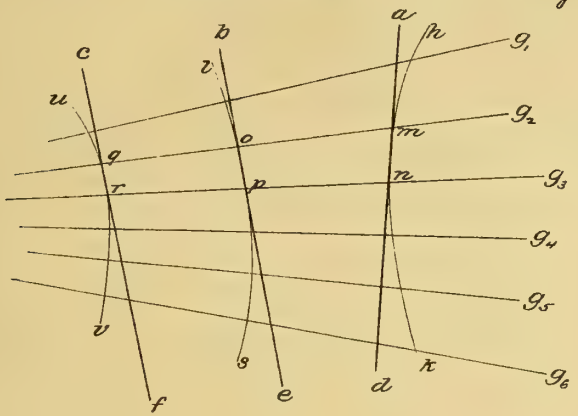


For, let aa' be any curve traced through a upon the curved surface. a and a' being consecutive points, a right line at through them will be tangent to the curve. By definition this tangent must lie in the given tangent plane, that is, in the plane of the elements oy and $o_1 y_1$.

Through any other points, b and c, on oy, let curves be traced at random on the given curved surface, as shown, intersecting $o_1 y_1$ in b' and c' respectively. The tangents bt_1 and ct_2 are clearly in the plane of the elements oy and $o_1 y_1$, that is, in the given tangent plane.

Thus a common plane contains all tangents to the given curved surface at points on the element oy, and is, therefore, tangent to the surface all along that element.

Theorem: A plane tangent to a warped surface at any point, is tangent at that point only, although it contains the rectilinear element passing through the point.



Let g_1, g_2, etc, be consecutive positions of the right line generatrix of a warped surface; and hk, rs, etc., any curves traced upon that surface. The right lines ad, be, etc., passed through the pairs of consecutive points m n, o p, etc., are tangent to the curves and to the warped surface.
Let a plane be tangent to the warped surface at o. It will contain the element g_2, and the tangent be, but no other tangent to the surface.

For, since the given surface is warped, the elements g_2 and g_3 are not in the same plane.
∴ the tangents be and ad are not in the same plane; neither are be and cf in the same plane.

It is clear, therefore, that the given tangent plane is tangent to the warped surface at one point only.

Theorem: In general, it is sufficient and necessary for the generation of a warped surface that a rectilinear generatrix rest constantly, in its motion, upon three curved directrices. (See figure on page 4).

Evidently three curves *might* be so chosen that the surface generated would be that of a cone or a cylinder; but, unless they are assumed in peculiar positions, with such end expressly in view, the surface will be warped.

For, let kk, ts, and rv be three curved directrices chosen at random.

Imagine two cones, having a common vertex at any point m on kk, and for their bases the curves ts and rv. These cones will, plainly, intersect along some right line mq, passing through the vertex m and resting upon the curves ts and rv. That line of intersection may represent one position of the rectilinear generatrix.

We may now imagine two other cones, having as before ts and rv for their bases, but having a common vertex at n, a point infinitely near m, on the curve kk. The right-line nr, in which these cones intersect, will represent the consecutive positions of the generatrix as it moves from m.

Proceeding in this manner, we may consider as many pairs of cones, and consequently as many positions of the generatrix, as we please, and the surface which is thus formed will be warped. For, when the generatrix passes from one position mq to the consecutive one nr, it may be assumed to glide along the three tangents ad, be, and cf; but those tangents are not in general in

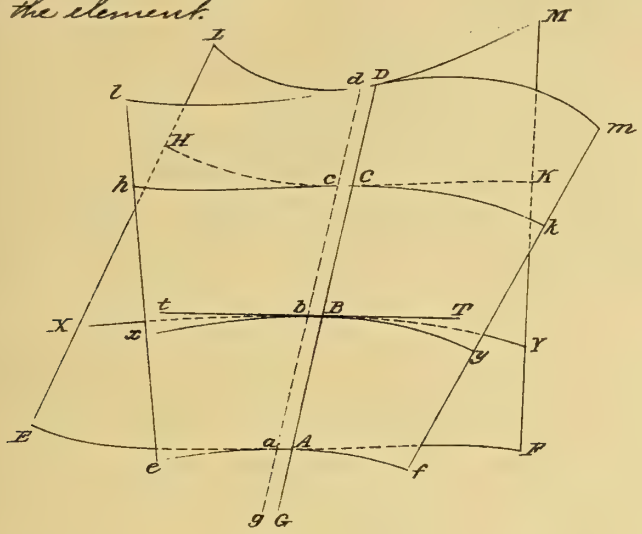
the same plane; consequently the consecutive positions mq and nr of the generatrix cannot be in the same plane; that is, the surface generated is warped.

Further, it is in general necessary that there should be three directing curves. If there were but two, the motion of the generatrix would not be properly determined for the generation of a warped surface; since, without changing its point of touching one of the directrices, it might be moved so as always to touch the other, in which case the surface generated would be, not warped, but conical.

If however, with only two directrices, the third condition be imposed that the generatrix shall, in its motion, always remain parallel to a given plane (called the "plane director") then the surface generated will be warped. For this third condition is equivalent to that of a third directrix lying in the plane director at an infinite distance.

Inasmuch as the right line is to be regarded as a curve of infinite radius, the theorem of this article may evidently be extended to include also the cases in which one, two, or all of the directrices are right lines.

Theorem: If two warped surfaces have a common rectilinear element, and are tangent to one another at three points of that element, then are they tangent all along the element.



Let two warped surfaces, S and S', have the common rectilinear element DG, and at three points, A, C, D, let them be tangent to each other; that is, at those points let them have common tangent-planes. Then will they be tangent all along DG.

For, let the surfaces be cut by any three auxiliary planes passing through the points A, C, D. Then will the curves EF, ef, cut from the warped surfaces by the auxiliary plane through A, be tangent to each other, since the surfaces have a common tangent-plane at that point. Similarly HK, hk, and LM, lm, will be, respectively, tangent to each other.

The curves EF, HK, LM, cut from S, may be assumed as directrices, moving along which the right line DG would generate the surface S. Similarly, ef, hk, lm, cut from S', may

be assumed as directrices, moving along which the right line DG would generate the surface S'.

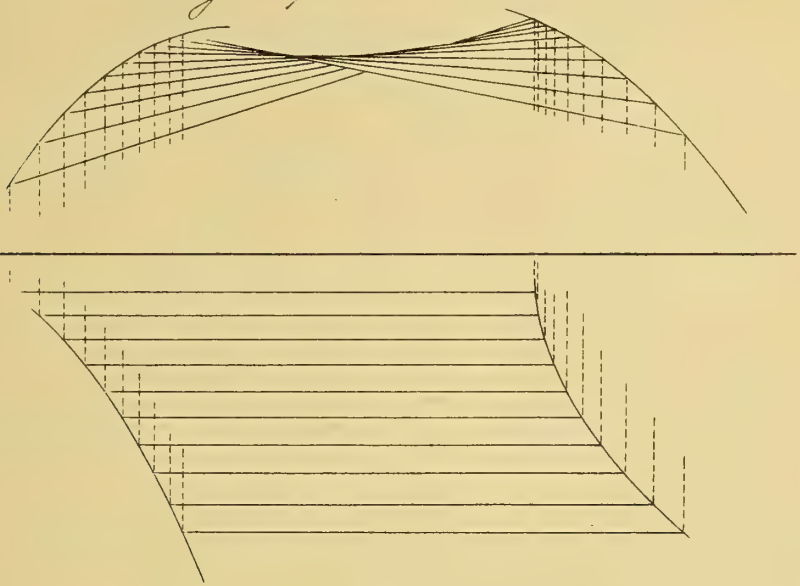
Suppose, now, the generatrix DG to move along either set of directrices to its consecutive position dg in the corresponding surface. It will at the same time continue to lie in the other surface also, since the three pairs of directrices, being respectively tangent to each other, have consecutive points in common. Since, therefore, the two warped surfaces have these two consecutive elements in common, it follows that they are tangent all along DG.

As further proof, suppose the given surfaces cut at the point B, chosen at random on DG, by any fourth plane, along curves XY, xy. Since these curves have two consecutive points in common, they have a common tangent tT. That tangent, however, must lie in a tangent plane to each surface at B, defined by the tangent itself and the element DG. Consequently, there is but one tangent plane to the two surfaces at B; in other words, they are tangent to each other at that point. But B being any point, the proof holds good for all points, and the theorem is established.

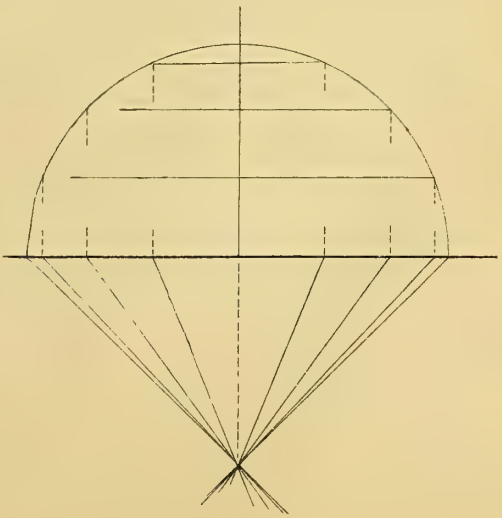
If the warped surfaces have a common plane director, then, in order that they be tangent all along a given element it is sufficient that they be tangent at <u>two</u> points on that element. For, as before, we may prove that the curves EF, ef, and LM, lm, passing through the points in question, are respectively tangent to each other. These two sets of curves, taken as directrices, together with a common plane director, are sufficient to insure that the generatrix in its motion from the position DG to the consecutive position dg shall remain at the same time in both surfaces. The remainder of the proof follows as before.

Among the great variety of warped surfaces which may be produced by imposing varying conditions as to the motion of the generatrix, may be noticed in particular:

The Cylindroid generated by a right line so moving as always to rest upon two given curves and at the same time remain parallel to a given plane.



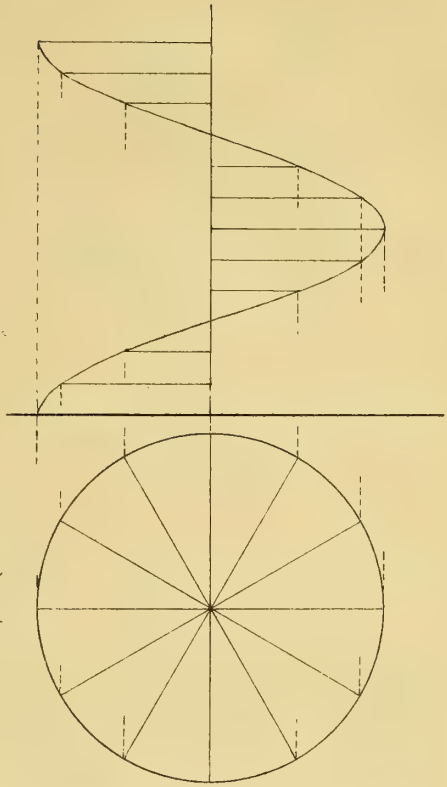
The Conoid generated by a right line so moving as always to rest upon a given curve and a given right line, and at the same time remain parallel to a given plane.



The Warped Helicoid with a Plane Directer, generated by a right line so moving as always to rest upon a helix and a given right line, and at the same time remain parallel to a given plane.

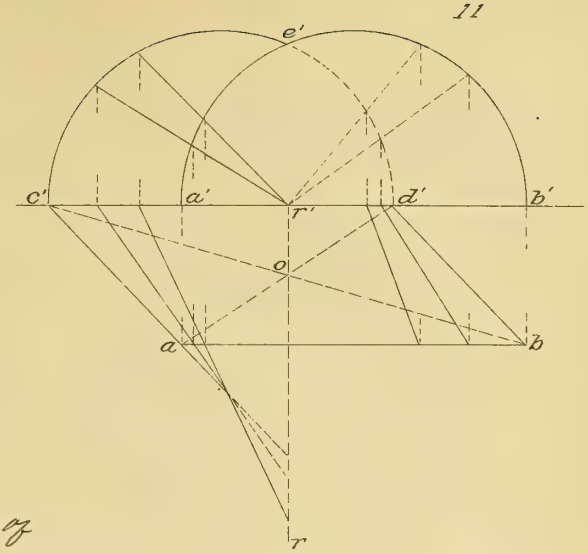
This is simply a particular case of the conoid.

Otherwise stated, this surface is one generated by a right line moving uniformly along another right line, and at the same time having a uniform angular motion about it and remaining constantly parallel to a given plane.



Other varieties of the warped helicoid may be generated by omitting the condition of a plane directer, and requiring the generatrix, in its motion about the rectilinear directrix, to make with it any constant angle other than a right angle.

The warped helicoid is illustrated in the curved surface of the ordinary screw thread.



The Cow's Horn (Corne de Vache), generated by a right line so moving as always to rest:

(1) upon a given circle (c'd', c'e'd') parallel to V.

(2). upon a second circle (ab, a'b'), equal and parallel to the first.

(3). upon a right line (rr') lying in H, perpendicular to V, and passing through the centre (o) of the parallelogram (abd'c').

If the position of the circles be such that the parallelogram become a rectangle, then the surface becomes that of a cylinder.

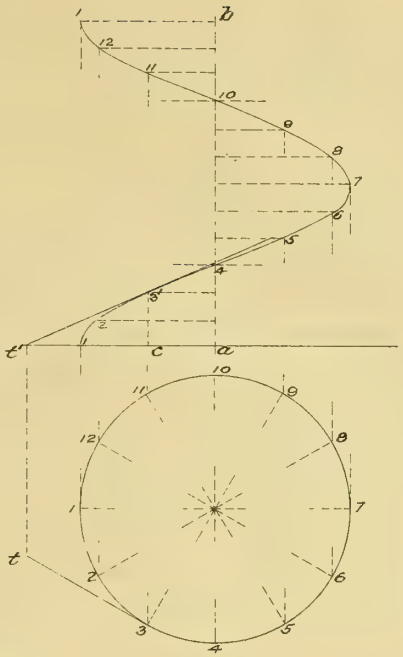
If the position of the circles be such that the parallelogram have its diagonal ad' perpendicular to V, then the surface becomes that of a cone.

The Hyperbolic Paraboloid, generated by a right line so moving as always to rest upon two other right lines, not in a common plane, and at the same time remain parallel to a given plane.

The intersection of this warped surface by a plane is either a hyperbola or a parabola.

The helix is a curve of double curvature, generated by a point moving uniformly around a right line (called the axis), at a constant distance from it, and with a uniform motion in the direction of the line.

The projection of the curve upon a plane perpendicular to the axis will evidently be a circle.



To construct the elevation of a helix, knowing the pitch ab, or distance which the generating point advances in the direction of the axis at each complete revolution:

Divide the circle 1-4-7-10-1 and the line ab into the same number of equal parts. Through the points of division of ab draw horizontals, upon which project from the successive points of division of the circle. If the points thus obtained be connected, the helix will be shown in elevation.

On account of the uniformity of motion, the ascent c-3 accomplished by the generating point in the direction of the axis, is directly proportional to the length of arc 1-2-3 traversed in plan. Hence, if we rectify any arc, as 1-2-3, and construct a right-angled triangle with this as a base, and c-3 as a side, the angle at the base will be the angle which the helix, or its tangent, makes with the horizontal plane. If, therefore, it is required to draw a tangent to the helix at any point, as 3, draw the

tangent 3-t, making it equal to the rectified arc 3-2-1. 3-t will be the plan, and t'-3' the elevation of the tangent, and t the point where it pierces H.

Since the helix and its tangents make the same constant angle, not only with the horizontal plane, but also with the elements of the imaginary cylinder upon which the helix is constructed, it follows that when the helix is unrolled upon a plane containing any tangent to it, the curve coincides with the tangent in the development. It also agrees with the tangent precisely in length, for any given interval parallel to the axis, since the length of each is determined by the hypothenuse of a right-angled triangle, whose base is a certain rectified arc, its side the corresponding ascent, and the angle at the base the constant angle made with the horizontal plane.

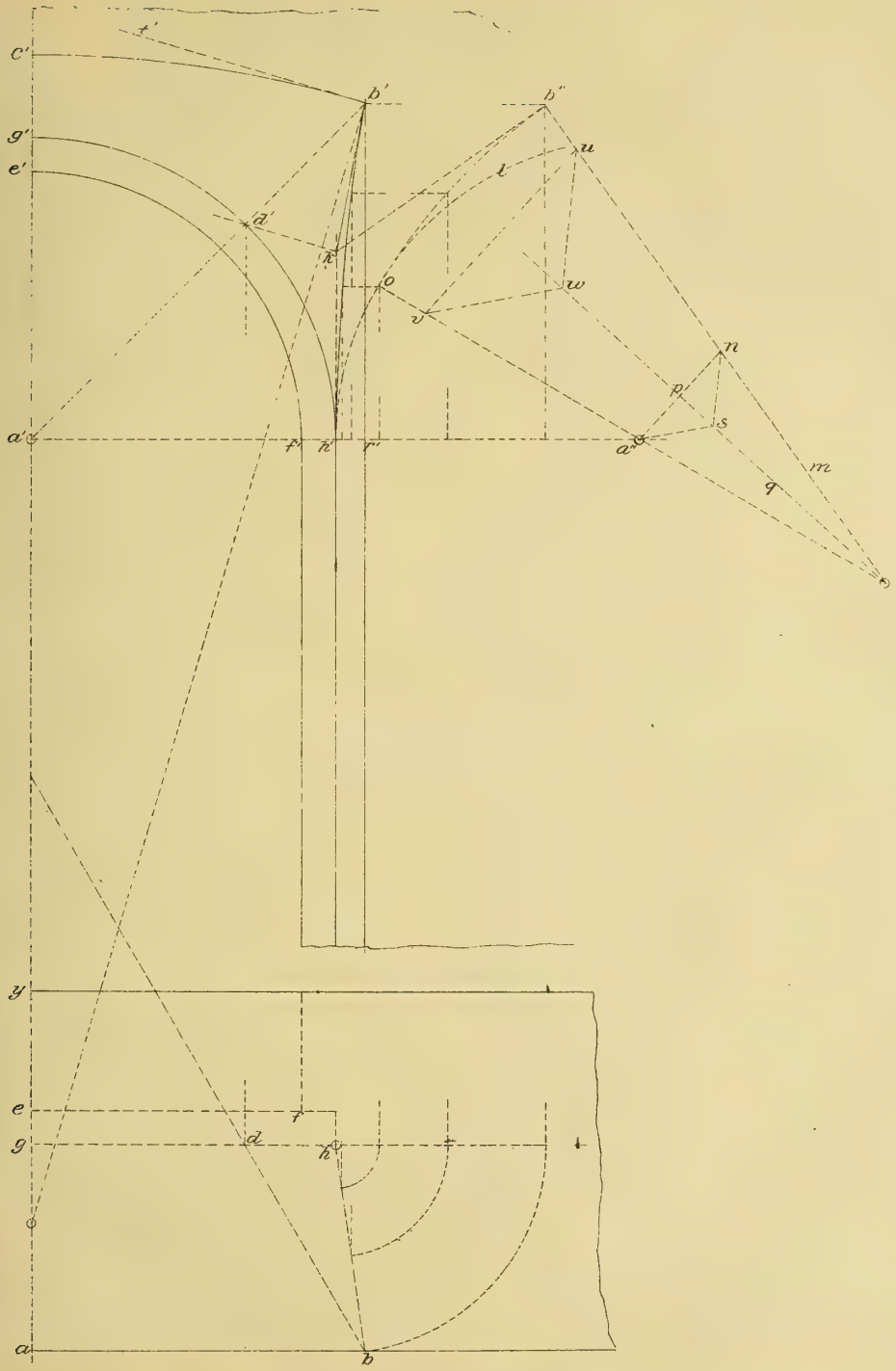
Gate Recess.

This construction is of an ornamental arched gate way, designed to accommodate the two swinging leaves of a gate. The most difficult part of the problem consists in arranging warped surfaces for the covering of the recess. One leaf of the gate is hinged at h. It closes against the annular surface included in elevation between the arcs e'f' and g'h', and opens against the side face of the recess projected in plan in bh. The height of opening, span, depth, etc., being supposed given, the arrangement of the surfaces is to be studied.

The covering of that part of the opening projected in plan within the rectangles he and fy is to be cylindrical.

Let the soffit of the outer part of the recess be a warped surface, with directrices as follows: (1) the axis of the arch. (2) the arc g'h'. (3) An arc b'c' arbitrarily chosen with regard to appearance. This arc is the line in which the warped surface is to meet the face of the wall, and should be so assumed that the point b' of meeting the edge of the recess shall be at least as high as g', the highest point of the gate. It would be found a disadvantage, in general, to continue this warped surface farther to one side than the element a'b', on account of interference with the opening of the gate.

The next step is to choose a second warped surface, which shall adjoin the one already assumed, and which, for the sake of appearance, shall be tangent to it along the element a'b'. As before, let the axis of the arch and


c'
g'
e'
a'
t'
b'
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m
q
f'
h'
f''
y
e
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h
a
b

the arc g'k' be taken as two of the directrices of the surface. Since the two warped surfaces have, therefore, a common element a'b', and two directrices in common, they have two common tangent planes, at a' and d', respectively. In order, however, that the two warped surfaces be tangent all along the element, they must have still a third tangent plane in common. That the third directrix of the second warped surface may be so chosen as to insure this, let the following construction be made:

At b' let a tangent b't' to the directing curve b'c' be drawn. This tangent and the element a'b' determine a plane tangent to the first warped surface at b'. If, through d' a parallel to b't' be drawn it will be the trace of the above tangent plane on a vertical plane through gk. This line intersects the vertical through k' at k. Let now the side face of the recess be revolved about the vertical k'k until parallel to V. The curve k'l shows the outline of the top of the gate when revolved, and b" is the revolved position of b'.

Let b" be joined to k, and let an arc of a circle be described which shall be tangent to b"k at b", and also tangent to the arc k'l. The construction for drawing the required arc is as follows: Draw b"m perpendicular to b"k. Take b"n = a"k' = a'k'. Join a" and n. Bisect a"n and erect the perpendicular pq. The intersection of nm and pq will be the center for the required arc, and a line from the center through a" will fix o, the point of tangency of the two arcs. Otherwise: Construct any triangle a"sn, with one vertex at a", one at n, and the third at any

point s on pg. Draw vv parallel to na"; vw parallel to ns; and wv parallel to sa". a"v produced defines o, the point of tangency, and the arc ob" can then be drawn by points. The compound curve b"ok' is to be the line of separation between the second warped surface and the side face of the recess, and is easily revolved back into its true position.

This curve is now to be taken as the third directrix of the second warped surface, and will insure the requisite third common tangent plane. For, a tangent plane to the second warped surface at b' will be determined by the element a'b' and the tangent b'k to the directrix just determined. Now these two lines pierce the vertical plane through gh at d' and k, respectively. ∴ d'k is the trace of a plane tangent to the second warped surface at b'. It has, however, previously been shown to be the trace of a plane tangent to the first warped surface at b'; hence the two surfaces have this tangent plane in common, and all the conditions have been established necessary for having them tangent all along the element b'd'.

The surfaces of the recess are now fully determined, and consist of:

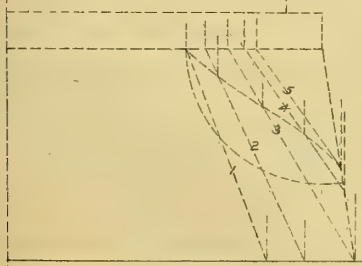
(a) A warped surface bounded by the curves d'g' and b'c', and the rectilinear element b'd'.

(b) A warped surface bounded by the curves b'k' and d'h', and the rectilinear element b'd'.

(c) A plane vertical side, bounded by the straight line b'r' and the curve b'k'.

The other half of the figure will correspond to that here shown.

In order to ascertain whether the surfaces as assumed would interfere with the free opening of the gate, a test may be applied as follows: When the gate is open its upper edge will generate a curved surface, and the problem is simply to find whether this surface intersects the warped surfaces within the limit of motion of the gate. This problem is to be solved in the usual manner, by cutting ~~both~~ surfaces by a series of horizontal planes, and observing whether the lines thus cut from the surfaces respectively intersect within the angle of motion. If they do, the arrangement of the surfaces must be modified. The construction is here shown for a single horizontal plane. 1, 2, 3, 4, 5, are rectilinear elements of the warped surfaces.



Helicoidal Oblique Arch

Principal Dimensions.

1. Angle of obliquity, or angle of skew, θ, = angle between springing line and face of arch.
2. Radius of cylinder (intrados) r.
3. Thickness of voussoirs, e, measured on radius perpendicular to axis.
4. External width of arch, b, measured perpendicularly between faces.
5. Span, measured at right angles to axis, = $2r$.
6. Oblique span, CD, measured parallel to face, = $2r \operatorname{cosec} \theta$.
7. Obliquity of arch, BD, = $2r \cot \theta$.
8. Length of arc of intrados in right section, = πr.
9. Intradosal angle, β. In development of intrados this is the acute angle between springing line and direction of coursing joints. $\tan \beta = \dfrac{\cot \theta}{\frac{1}{2}\pi}$
10. Extradosal angle, ϕ. Corresponds to intradosal angle.
$$\tan \phi = \frac{r+e}{r} \tan \beta.$$
11. Twisting rule angle, δ, = $\phi - \beta$. Measure of the warp.
12. Length of heading spiral, DC", = $\pi r \sec \beta$.
13. Number of voussoirs, n, (assumed).
14. Width of each voussoir, = length of heading spiral divided by number of voussoirs, = $\dfrac{\pi r \sec \beta}{n}$.
15. Length of impost, DN, = $b \operatorname{cosec} \theta$.
16. Divergence of courses, NQ, = $b \operatorname{cosec} \theta \sin \beta$.
17. Theoretic eccentricity (of intrados) = $\dfrac{r \cot^2 \theta}{\frac{1}{2}\pi}$. Determines the direction of the joint-lines on the face of the arch.

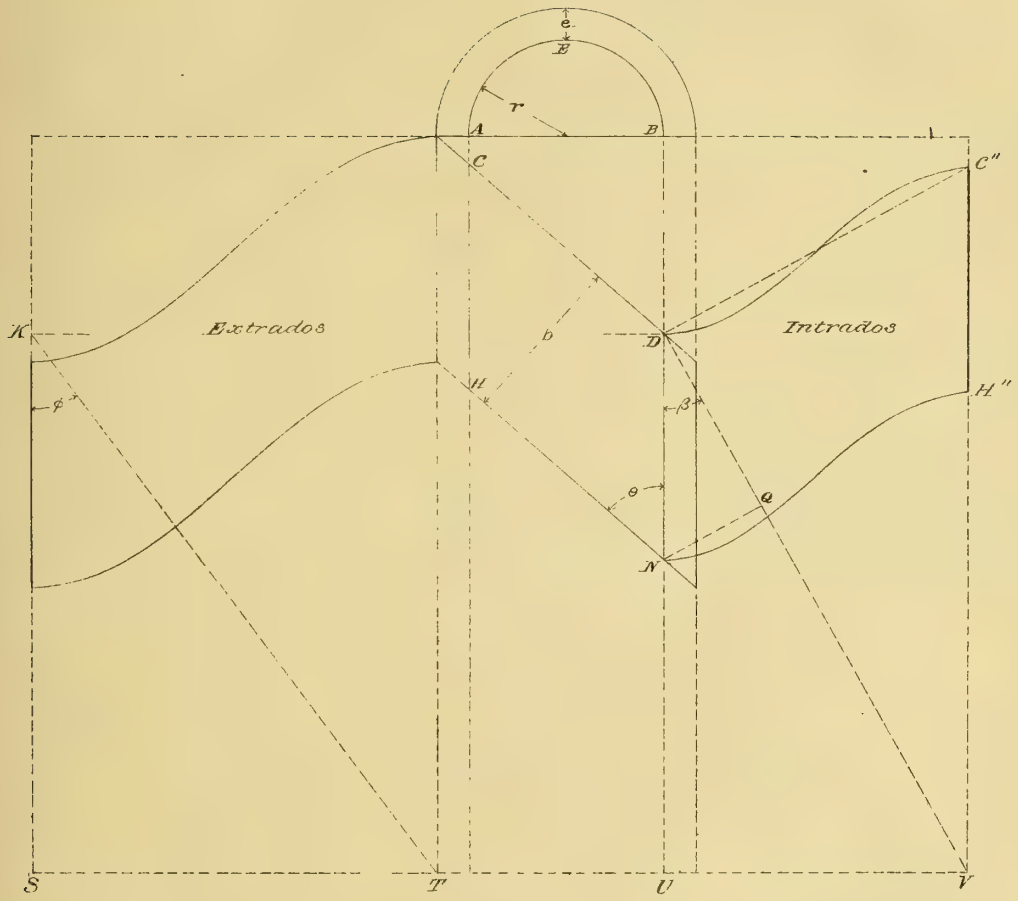


Fig. 1.

Of the various dimensions that have been enumerated, the first five are arbitrarily assumed; from those may be obtained the others, either graphically or by calculation. In making the drawings, sundry problems arise which will be considered:

(a). **Development of intrados.** This surface, being cylindrical, is developable, the springing lines remaining straight in development, and the elliptical ends becoming reversed curves. Any given point, as m, can readily be located by projection, as at m", the distance $Bm'' = $ the rectified arc BBm.

The development can be effected independently of projection, by the use of computed co-ordinates, the best method when a full-size drawing is to be made. The relative position of the points N, D, C", H" (Fig. 1) in development can easily be computed, and the parallelogram drawn. It remains to determine coordinates of points on the reversed curves. It will be noticed that those curves are precisely alike, and that each consists of halves identical in shape, but reversed in position; it is only necessary, therefore, to determine coordinates for a half curve. Let it be required to find in development the coordinates, x', y', of any point e.

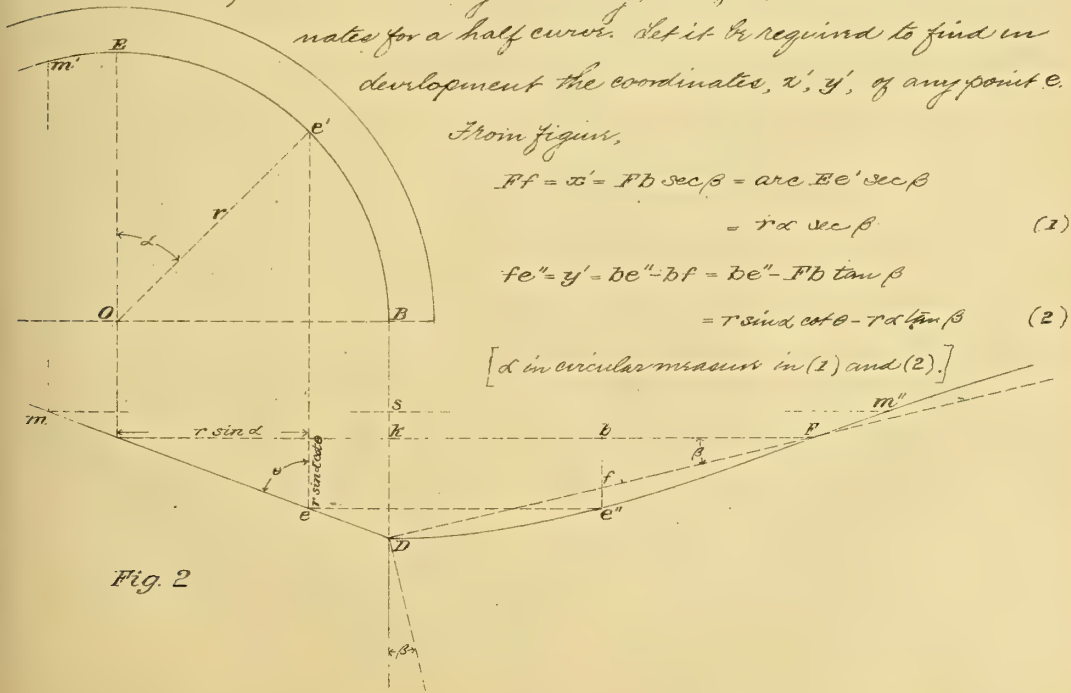


Fig. 2

From figure,

$$Ff = x' = Fb \sec\beta = \text{arc } Be' \sec\beta$$

$$= r\alpha \sec\beta \qquad (1)$$

$$fe'' = y' = be'' - bf = be'' - Fb \tan\beta$$

$$= r \sin\alpha \cot\theta - r\alpha \tan\beta \qquad (2)$$

[α in circular measure in (1) and (2).]

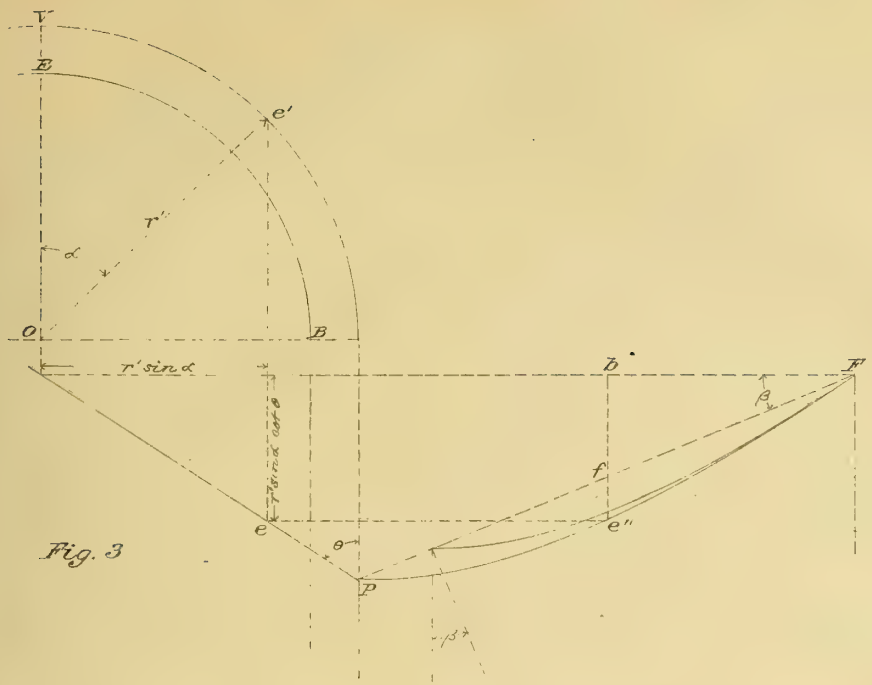


Fig. 3

(b). Development of extrados. The extrados has, theoretically, a cylindrical surface, the development of which may be made, when required, after precisely the same method employed for the intrados. Thus, from the figure, which represents both intrados and extrados developed with reference to the centre line through F,

$$Ff = x' = Fb \sec\beta = \text{arc } Vc' \sec\beta = r'\alpha \sec\beta \qquad (3)$$

$$fe'' = y' = be'' - bf = r' \sin\alpha \cot\theta - Fb \tan\beta$$

$$= r' \sin\alpha \cot\theta - r'\alpha \tan\beta \qquad (4)$$

[α in circular measure in (3) and (4).]

In general, however, the actual surface of the extrados is rough, and the development, even of the theoretical surface, is not really necessary.

(C) Arrangement of Joints. (Fig. 4).

(1) Intrados. Assuming the thrust of the arch transmitted to the abutments in lines parallel to the faces of the arch, the curve D"C" shows the proper direction for heading joints in development of intrados. As a convenient and suitable approximation, however, the straight line D"C" is taken as a guide in place of the curve, and the coursing joints are drawn perpendicular, or nearly so, to this line. They are thus perpendicular to the various transverse or heading joints, and approximately so to the face lines of the intrados. Both of the right lines D"C", N"H", are to be divided into the same odd number of equal parts, and through the points of division of D"C" perpendiculars are to be drawn. Preferably these should meet N"H" at opposite points of division. If they do not, either :—

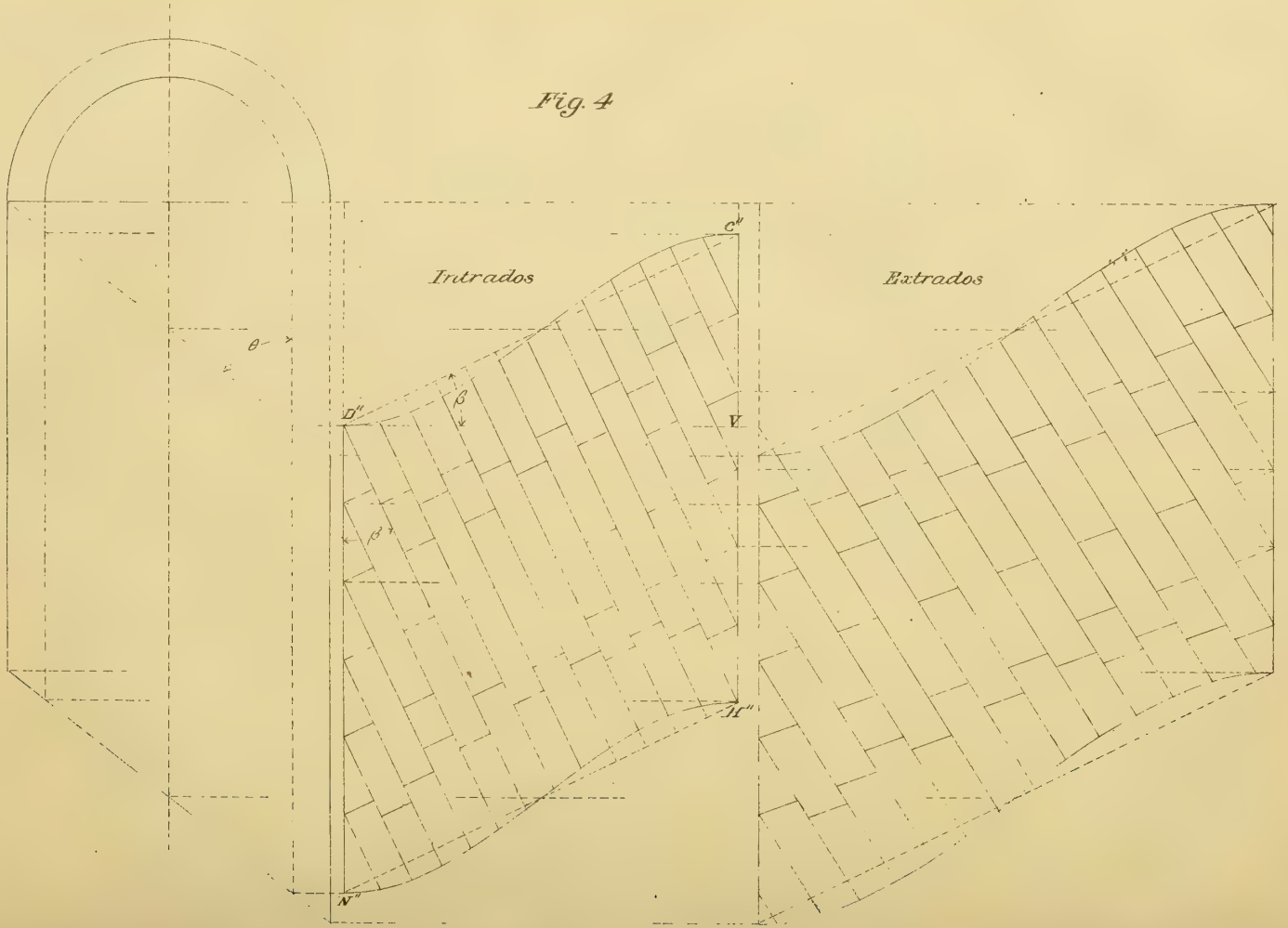
(a) The coursing joints may be drawn between nearly opposite points of division of D"C" and N"H", without regard to being strictly perpendicular to either line, or

(b). The proportions of the structure may be slightly changed, so that the coursing joints shall pass through opposite points of division, and at the same be perpendicular to D"C" and N"H".

The heading joints are now drawn at right angles to the coursing joints, being arranged according to judgment so as to divide the arch into voussoirs.

(2) Extrados. The arrangement of joint lines in the development of the extrados must be consistent with that already determined upon for the intrados. In making the drawing it is only necessary to bear in mind that the line which generates the warped or joint surface is constantly at right angles to the axis of the arch, and that it simultaneously traces the joint lines on both intrados and extrados. It is evident from

Fig. 4



the figure that if, in the development of the intrados, the two sets of joint lines are perpendicular to each other, they cannot be so in the development of the extrados. For this reason there would be some theoretical advantage in making the joint surfaces normal at a point intermediate between intrados and extrados. This can be effected by so arranging the proportions of the structure, or the division into voussoirs, that the intradosal angle β, as shown in Fig. 4, shall be somewhat decreased. The coursing and heading joint lines will not then be normal to each other in either intradosal or extradosal development; but they would be normal in a development made at some intermediate point.

From Fig. 4,

(Intradosal angle) $\tan\beta = \frac{VC''}{VD''} = \frac{2r\cot\theta}{\pi r} = \frac{\cot\theta}{\frac{1}{2}\pi}$.

From Fig. 1,

(Extradosal angle) $\tan\phi = \frac{ST}{SK}$. Also, $\tan\beta = \frac{UV}{UD}$.

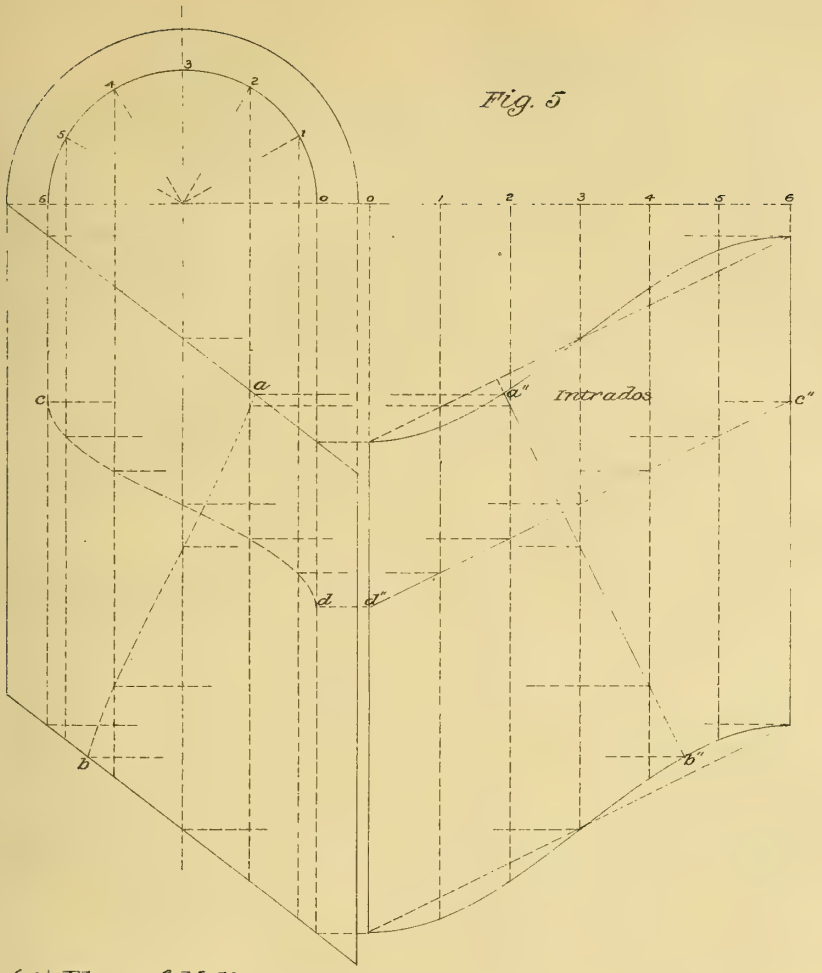
$$\therefore \tan\phi = \frac{ST}{SK}\cdot\frac{UD}{UV}\cdot\tan\beta \text{ ; or, since } UD = SK,$$

$$= \frac{ST}{UV}\cdot\tan\beta.$$

$$= \frac{\pi(r+e)}{\pi r}\cdot\tan\beta.$$

$$= \frac{r+e}{r}\cdot\tan\beta.$$

Fig. 5



(d) Plan of Helices.

It is clear that the joint lines, which in development are straight, will, when the intrados and extrados are turned back into their true positions in the arch, become spiral curves. They will, in fact, be helices. The plan of any of these curves may easily be determined by projecting across from the development, as shown in Fig. 5, it being simply necessary to remember that any given point will lie on the same rectilinear element (of the cylinder) in development and in plan.

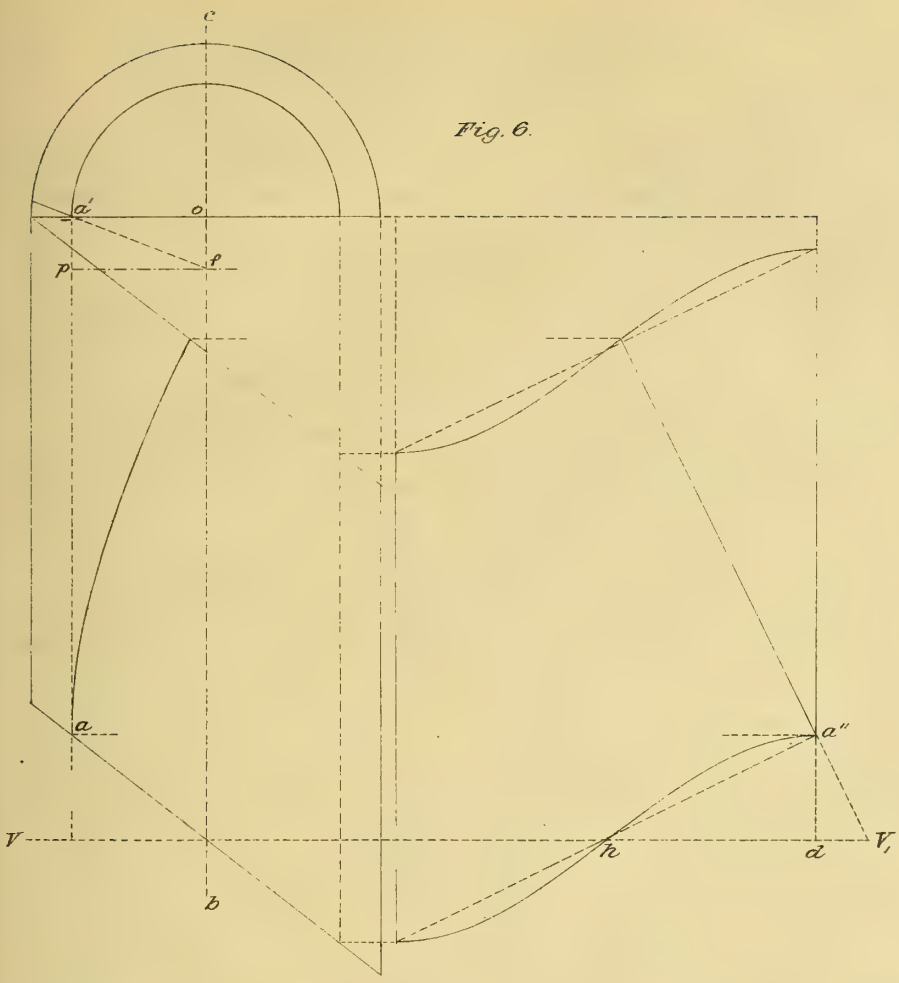


Fig. 6.

(c) *Focus and Eccentricity.*

A little thought will show that the radiating joint-lines which appear on the faces of the arch should, strictly, be curved, being the lines in which the plane face of the arch intersects the warped coursing-joint surfaces.

From the intradosal development, by aid of projection, one extremity of each curve can be obtained. From the extradosal development, similarly, the other extremity can be determined, and from intermediate developments any desired

number of additional points on the curve.

Otherwise, we may intersect both helicoidal surface and the plane of the face of the arch by auxiliary planes passing through the axis. Each of these planes will cut a rectilinear element from the helicoid, and a straight line from the plane, the intersection of the two lines furnishing one point in the required curve.

Still again, tangents to the curve at various points can be determined, and the curve drawn with these as guides. The curvature is in general slight, however, and in practice is usually disregarded and the curve taken as coinciding with the tangent drawn at the intrados, or in some cases with the tangent drawn at the extrados. All such tangents at the intrados meet at a common point, called the focus. Similarly, all tangents to the face-joint curves at the extrados have a common focus; and the same is true for any other set of tangents at a given distance between intrados and extrados.

Let it be required to draw a tangent to the joint curve at a (Fig. 6):

Let S denote helicoidal coursing-joint surface which starts from a.

Let P_a denote plane of arch face.

" P_v " vertical plane through V–V.

" P_t " plane tangent to S at a.

The joint curve is the intersection of P_a and S. A tangent to it will therefore lie in P_a and in P_t; it will pass through a, and will meet BC (the line of intersection of P_a and P_v) in a point to be determined.

P_c will be determined by the element ao and a tangent to the coursing-joint helix at a. This tangent will evidently prime P_v at p, the distance a'p being equal to dV_1.

pf, parallel to the element ao, is then the intersection of P_c and P_v. Now the required tangent lies in P_a and in P_c; it must, therefore, intersect P_v at some point common to the three planes. f is the only common point, and fa' is the tangent sought. The distance of is called the eccentricity of the arch, or perhaps more properly the eccentricity of the intrados.

From Fig. 6,

$$of = dV_1 = a'd \tan\beta = hd \tan^2\beta$$

$$= \frac{\pi r}{2} \tan^2\beta$$

$$= \frac{r \cot^2\theta}{\frac{\pi}{2}} \quad \text{(for intrados)}$$

Similarly, for extrados, eccentricity $= \frac{(r+e)\cot^2\theta}{\frac{\pi}{2}}$

$$= r \cot\theta \tan\phi.$$

(f.) Elevations of Arch. An elevation of the arch as viewed at right angles to the face may be required.

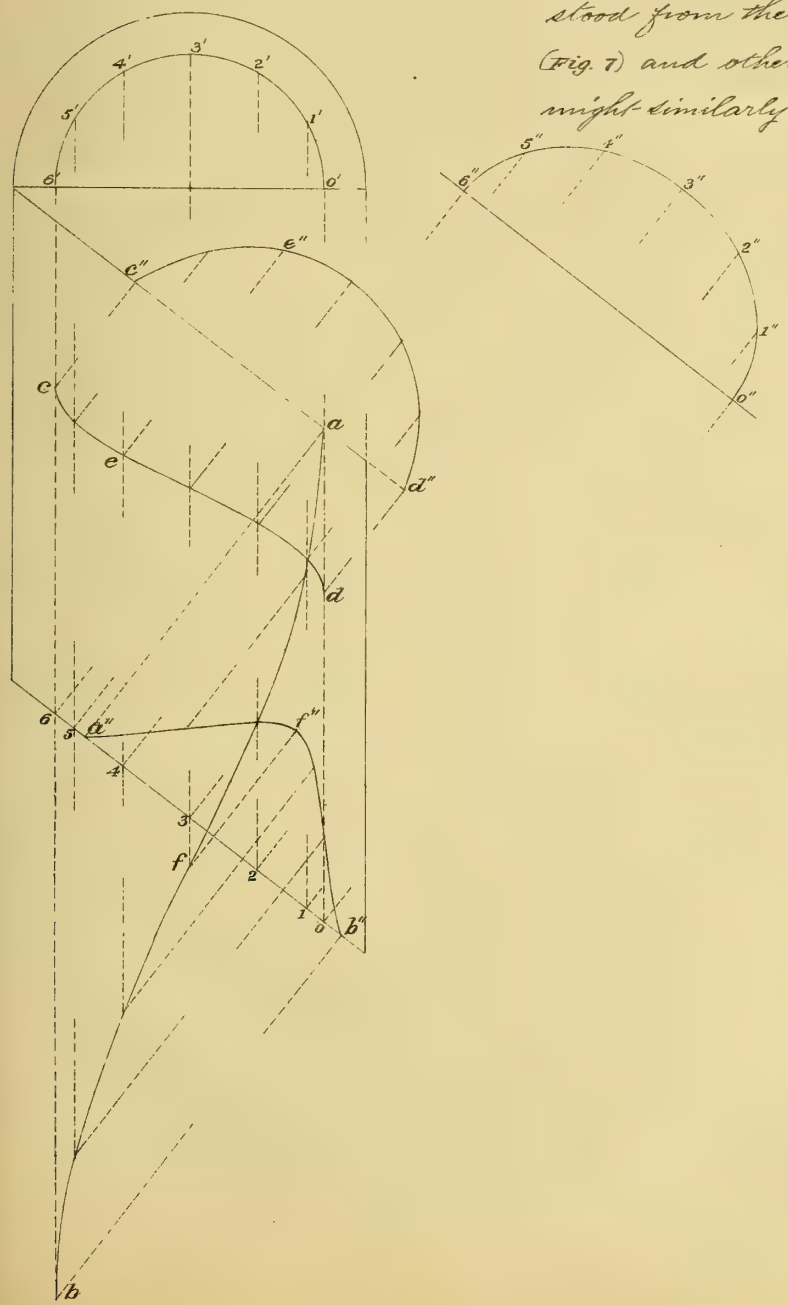
In such view the face lines of intrados and (theoretical) extrados appear as ellipses, which may be determined either by projection or by independent construction, the span and rise of the curves being known.

The coursing and heading joint lines may also be determined by projecting separately for each one; or better by making complete projection of a single spiral of each set, thus obtaining a template curve which may be used successively in drawing

all the other joint lines of the set.

The mode of determining these various curves by projection will readily be understood from the drawing (Fig. 7) and other elevations might similarly be constructed.

Fig. 7.



(9) Sheeting Voussoirs. In the helicoidal arch, all voussoirs except the ring stones may be exactly alike. Either bricks or stone may, therefor, be used. If bricks be the material, the ring stones will be given a width on the intrados corresponding to a definite number of courses of bricks. If stone be employed, the width will not be limited by such consideration. The sheeting voussoirs may be cut to shape by the methods to be described. Each stone will have two warped coursing-joint surfaces, two warped heading joint surfaces, a cylindrical intrados, and theoretically a cylindrical, but practically in general a rough, extrados.

Having selected a rough block of sufficient size to contain the finished stone, a convenient order of working will be to dress, first, a coursing joint surface; next, the intrados; then, the other coursing joint surface; and, finally, the heading joint surfaces.

A coursing joint surface can be cut by first chiseling drafts to correspond to two or more elements, and then dressing off the intervening material to a surface determined by a straight-edge resting upon the two elements as directrices and held parallel to the soffit edge of the joint surface. By this method, as close an approximation as desired to the theoretical spiral or winding surface of the joint can be obtained. For cutting the drafts which represent elements, "twisting rules" are used. Two rules, one with parallel edges, and one with converging edges, are commonly sufficient; but for long stones, or an extreme warp, additional tapering rules can be employed, more elements thus being located. The rules should be so joined by rods as to maintain their proper relative position.

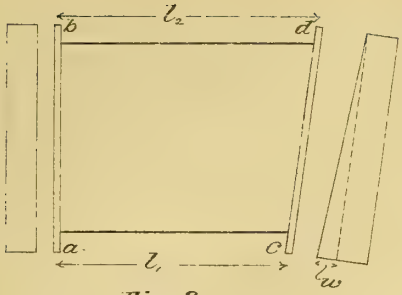


Fig. 8.

The length ab = cd = radial thickness of arch.
The length l_1 is arbitrary.
The length $l_2 = \frac{l_1 \sec \phi \cos \delta}{\sec \beta}$
The length w, which measures the warp of the surface, $= l_2 \tan \delta$
The values l_2 and w may be deduced as follows:

Let Pmn (Fig 9) be the portion of a helicoidal coursing-joint surface included between the vertical element P and the inclined element mn. Let tt_1 be the H trace of any vertical plane parallel to Pm. Let the elements P and mn be projected perpendicularly upon that plane. Then it is evident that the two elements, with their projecting planes, correspond to a pair of twisting rules, placed with two corresponding edges "out of wind" in the plane tt_1, the other two edges being elements in the warped surface.

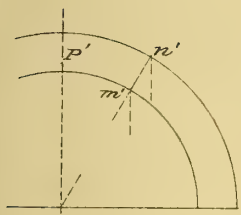
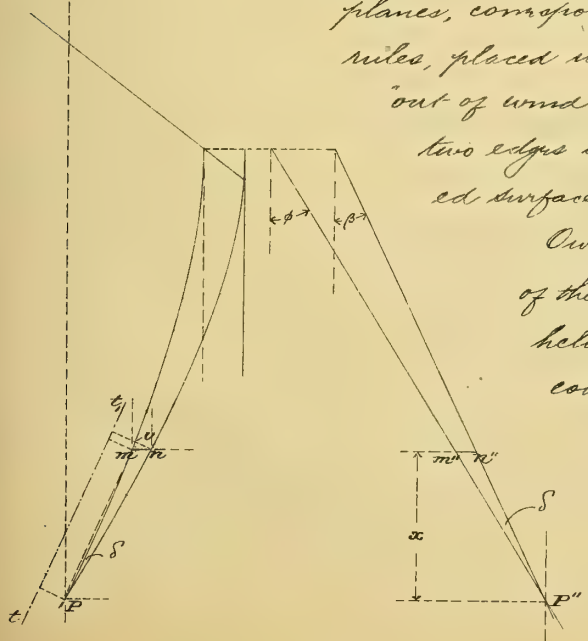


Fig. 9.



Owing to the uniform motion of the line which generates the helicoide, these twisting rules could be applied equally well in fixing elements upon any other equal length of the coursing-joint surface.

For the length of an ordinary voussoir, the chords Pm and Pn of the intradosal and extradosal helices, respectively, will not differ materially in length from the curves themselves, the true lengths of which are shown in the developments.

$$\therefore l_1 = Pm = x \sec\beta$$

$$l_2 = Pn \cos\delta = x \sec\phi \cos\delta$$

$$\therefore l_2 = l_1 \frac{\sec\phi \cos\delta}{\sec\beta}$$

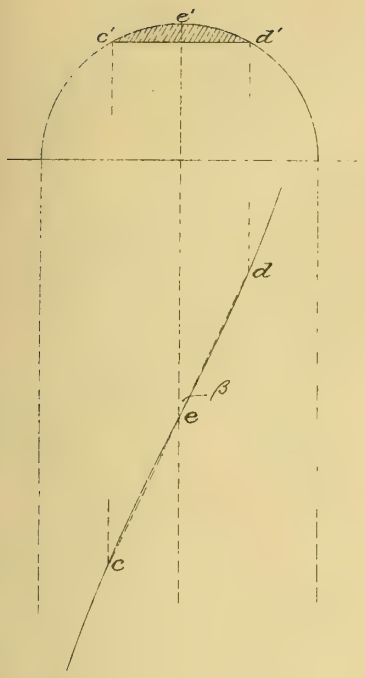
The angle $\delta\ (=\phi-\beta)$ is called the angle of twist, and is, plainly, determined by the rate of advance of the generatrix of the warped surface, and by the thickness of the arch. The distance $vn = w$ measures the warp of the helicoid from the vertical plane Pmv, in the distance here considered. From the figure, $w = (\text{approximately})\ l_2 \tan\delta$.

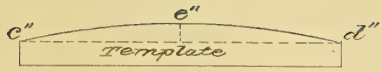
By aid of the figure more precise values can be deduced for l_2 and w, if needed.

In dressing the stone, care must be taken that the warp be given the right direction, i.e., right-handed or left-handed, as the case may be.

Coursing Joint Template. In indefinite surface of a coursing joint having been worked, the intradosal edge may be determined by applying a curved template constructed as follows (see Fig. 10): The length ab is arbitrary, but should be as great as the length of a voussoir. Within the limits of such length the intradosal edge may be regarded as a plane curve, approximating to a circular arc, and, more closely, to the arc of an ellipse. cd may then be considered the plan, and c'd' the elevation, of the template. The chord

Fig. 10.





c'd' will equal cd sin β, the ordinate at e' may readily be calculated, and the required curve will be either a circular arc passing through the points c, e, and d; or, more nearly, the arc of an ellipse passing through the same three points, ordinates to which curve may be found either by projection or by computation.

Soffit Frame. The intrados or soffit is next to be cut, and for this purpose a "soffit frame" or "saddle mould", the construction of which is shown in Fig. 11, may be used to advantage. ac and df are templates with one edge cut to the curve of a right section of the intrados. The arms at the end have their edges ce and df normal to the curves, that is, corresponding to elements of the joint surface, and are of length equal to the depth of the voussoirs. The templates are to be accurately and rigidly framed together in the position shown, with the angle adc = β. The stone having been placed with soffit side uppermost, the soffit frame is inverted and set against the stone as shown in Fig. 12. The radial edges of the arms now take the position of elements of the helicoidal joint surface already cut. Draw a line along

Fig. 11

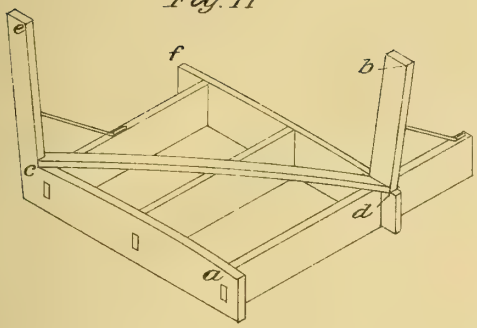
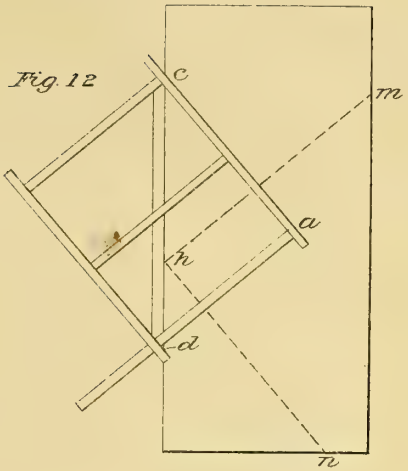


Fig. 12



ad, which is parallel to axis of cylinder, and another along ac at right angles to axis. Then sink chisel drafts along these lines until the soffit frame fits well along all the bearing edges.

"Two segmental pieces, each as long as ca and of same radius, are used, one to apply to draft ac and the other upon a line nn drawn at a convenient distance and parallel, with their centres on the line nn. The second segment is then sunk in a chisel draft until its upper edge is out of wind with the first segment. This being done, the superfluous stone on the soffit is dressed off until a straight-edge applied parallel to ad coincides with the soffit in all parts."

The other soffit-coursing-joint edge may now be cut by ganging across from the one already cut, or a soffit-pattern may be applied, and the entire outline of the soffit traced.

The second coursing joint surface may next be worked by turning the soffit frame about, and reversing the process applied to cutting the soffit.

Finally, the heading-joint surfaces may easily be obtained by aid of the edges already determined.

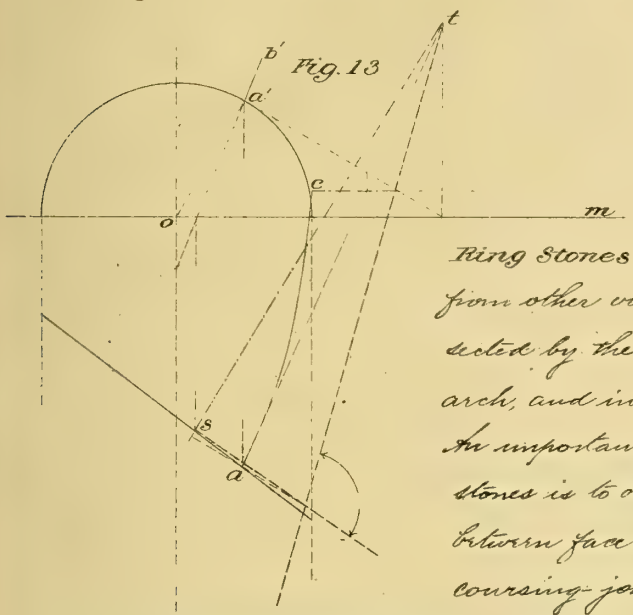


Fig. 13

Ring Stones. The ring stones differ from other voussoirs in being intersected by the plane of the face of the arch, and in being irregular in length. An important feature in cutting these stones is to obtain accurately the angle between face and soffit edges of coursing-joint surfaces. The angle may be obtained graphically as follows: Let a'b' (Fig. 13) be a face-joint line, and ca the intradosal helix passing through a. Construct a tangent to the helix at a, intersecting H at t. The face joint-line b'a' intersects H at s. Revolving the two lines sa and ta about the trace ts, into H, the required angle will be shown in its true size.

Another and an extremely simple method of finding the angle by construction is that of M. de La Gournerie, as follows (see Fig. 14):

Lay off C'O' = eccentricity = $r \cot\theta \tan\beta$

" " O'Q' = $r \cot\theta \sec\beta$

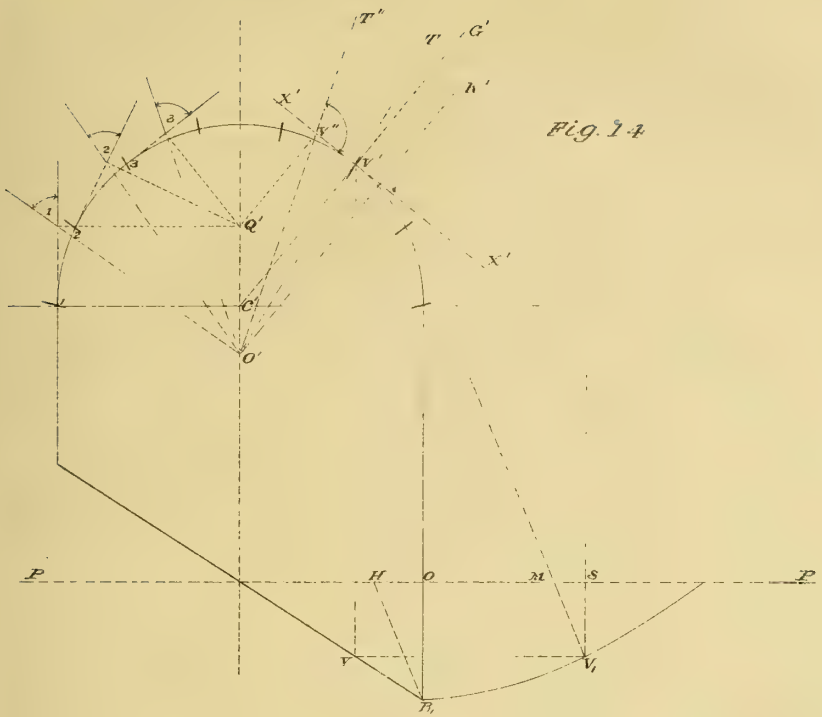


Fig. 14

At any point in question, as V', draw a tangent X'X', let fall the perpendicular Q'V'', and produce O'V''. Then is T''V''V' the required angle.

For: the angle to be found is evidently that between a tangent to the coursing joint helix, where that curve meets the face, and a tangent to the face line of the joint surface at the same point.

The vertical projection of the tangent to the helix will be X'X', and of the tangent to the face line, T'V'.

The element of the warped surface passing through V' must be perpendicular to the tangent X'X', since X'X' is in a plane tangent to the cylinder at V', and the element is normal to the cylinder.

The element also lies in the plane of the tangents $X'\dot{X}'$ and $T'O'$; for those lines, being respectively tangent at V' to two curves lying in the warped surface, must themselves lie in a plane tangent to the warped surface at V', which tangent plane must also contain the element passing through the point of tangency.

The element is parallel to the vertical plane of projection; consequently the trace of the tangent plane upon the vertical plane PP will be parallel to the vertical projection, $C'G'$, of the element, and will be a line $O'K'$ passing through O', the point at which the tangent $T'V'$ pierces the plane PP.

Now suppose the tangent plane to revolve about its trace, $O'K'$, until parallel to the vertical plane of projection. The tangent to the helix will still be projected along $X'\dot{X}'$, and the point V' will fall at V''. The length $V''R'$, the true distance of V' from the axis of rotation, is plainly equal to the length of tangent between V' and the plane PP; it must also equal the distance V,M, since in development the helix and its tangent coincide.

The point O' of the tangent $O'V'T'$ remaining fixed, the revolved position of the tangent will be $O'V''T''$, and the angle $T''V''V'$ will be the angle sought.

If now, from V'' we draw $V''Q'$ perpendicular to $X'\dot{X}'$, and therefore parallel to $C'G'$ etc., it will meet the perpendicular through C' at a point Q', which enjoys the remarkable property that through it will pass all other perpendiculars obtained similarly to $V''Q'$.

For, since $V,M = V''R'$, SM (the projection of V,M) $= V'R'$ (the projection of $V''R'$ before revolution). Further, it is seen that $O'Q'$ and $X'\dot{X}'$ are cut by three parallel lines into proportional

Logarithms. $\frac{O'C'}{O'Q'} = \frac{R'V'}{R'V''} = \frac{SM}{V,M} = \frac{OH}{B,H}$ (by similar triangles)

$\therefore O'C' \times B,H = O'Q' \times OH$

But $O'C' = OH$ Hence $B,H = O'Q'$, and the position of Q' is thereby fixed.

The above result being evidently independent of the particular position V' of the point in question, it appears that the position of the point Q' may be established in advance, and by its aid all the various angles, 1, 2, 3, etc., may readily be obtained in their true size.

It will be observed that at equal heights the angles on opposite sides of the centre are supplementary.

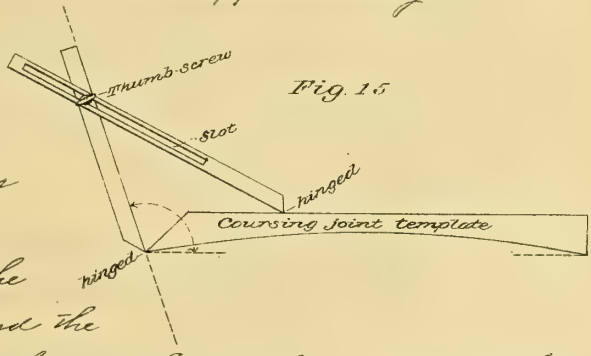


Fig. 15

The angles may be found mechanically as follows:

The centering for the arch being in place, and the joints for joint lines of the face having been marked by a nail, stretch a cord from the focus to correspond to any given joint line. At the same time apply an intradosal coursing joint template, made as shown in Fig. 15, to the lagging, along which the intradosal coursing helices have previously been traced. The template may be made in the form of a shifting-block, and the blade clamped so that its edge shall coincide with the cord stretched from the focus. The required angle is thus obtained directly.

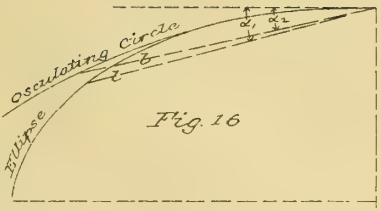


Fig. 16

If, for convenience in subsequent use, the angle sought be that between face line and chord of intradosal edge, rather than that between face line and tangent to intradosal edge, then the angle obtained as previously explained must be increased by the angle between tangent and chord. This angle may be obtained either graphically or by computation, as follows (see Fig. 16):

Let l = length of chord, arbitrarily chosen. Assuming the intradosal curve to be an ellipse, the semi-minor axis will equal r, the radius of the arch; the semi-major axis $r \operatorname{cosec} \beta$, and the required angle $\alpha_1 = \sin^{-1} \frac{b-y}{l}$.

Assuming the intradosal curve to be an arc of the osculating circle of the ellipse, the radius of curvature $R = \frac{a^2}{b} = \frac{r}{\sin^2 \beta}$, and $\alpha_2 = \sin^{-1} \frac{l}{2R} = \sin^{-1} \frac{l \sin^2 \beta}{2r}$.

The curvature being constant along the spirals, the correction angle will be the same at all points.

The order and mode of dressing the surfaces of a ring stone will in general be the same as already outlined for a sheeting voussoir, except that after cutting coursing-joint beds and soffit, the shifting stock, set to the proper angles as above determined, must be applied to fix the radiating edges of the face, the surface of which may then easily be dressed off.

If the ring stones are not cut with a cylindrical

extrados, the top and side must be squared back from the face to receive the adjacent spandrel face stones, or "creepers".

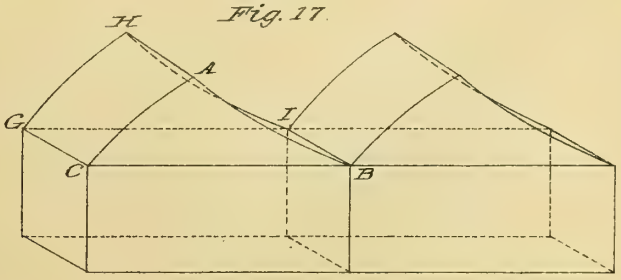
Skewbacks. If the arch is to be entirely of stone, the skewbacks forming the impost are carried down one course into the abutment, so as to avoid acute angles in the stone, and danger of sliding; and are also carried up into the arch far enough to present a heading-joint surface to one of the courses of shelling voussoirs. There will thus be a series of checks or "saw-teeth". If the shelling of the arch be of brick, the skewbacks will yet be cut with checks of such width as to intercept either single courses of bricks, or some other definite number of courses.

Let the skewback be of the form shown in Fig. 17:

Below the plane CI from which the arch springs the stone will be bounded by plane surfaces. This portion will be cut first, but the mode of dressing need not be described. Above the springing line the intrados may then be worked off by aid of a template shaped to the curve of right section, and a pattern of the intradosal surface of the stone. The outline CAB will thus be obtained.

If the extrados is to be cut to a cylindrical surface, this may next be obtained in the same manner as the intrados. For the warped surface CH, the edges CA and GH may now be taken as directrices,

Fig. 17.



divided both into the same number of equal parts, and the surface reduced by aid of a straight-edge applied to corresponding points of division.

If the extrados is to be rough, then the normals CG and AH to the intrados may be used as directrices for the surface CH.

The same method applicable to one warped surface may be used for both.

Tendency of Voussoirs to Slide. The winding of the coursing spirals about the barrel of a full-centered arch may give the coursing joints such an inclination to the horizon, that the lower ring stones will tend to slide out. This tendency may be provided against by arranging the wing walls to project somewhat in front of the lower ring stones, and thus receive their thrust.

Laying out Spirals on Arch Centring. The spirals should be marked on the lagging of the centring, as guides in setting the voussoirs in place during construction. This may be done as follows (see Fig. 18):

From the figure, $\gamma = \tan^{-1} \frac{2}{\pi \cot \beta}$

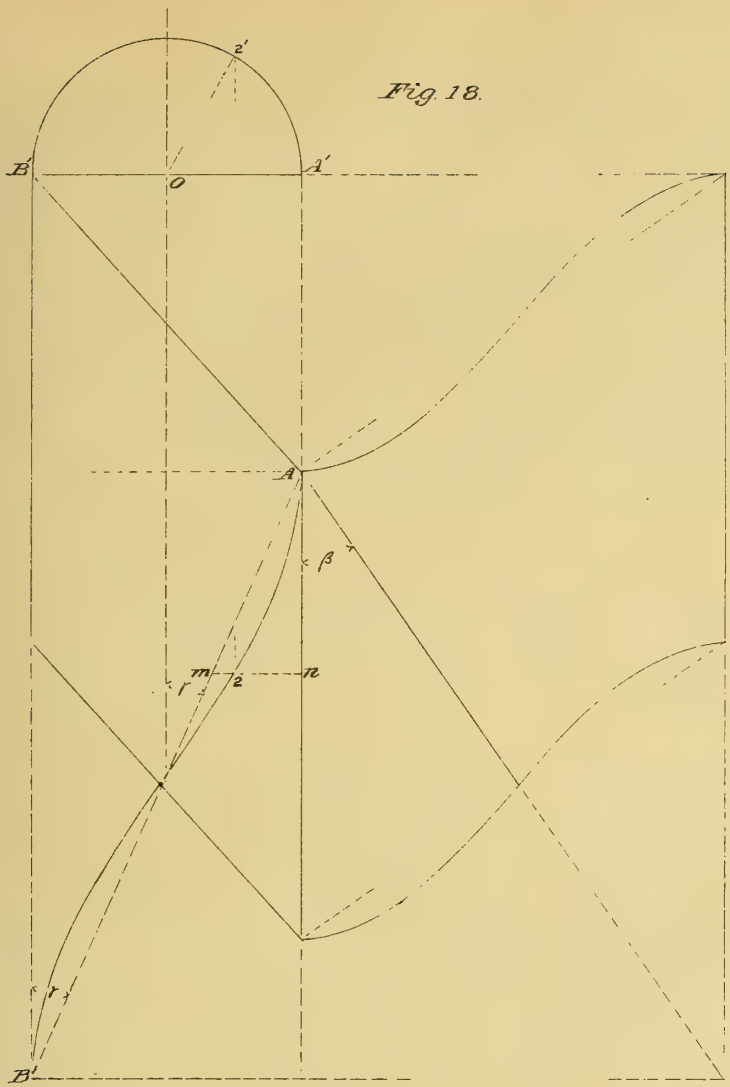
$$Am = \frac{A'2'}{A'2'B'} \cdot \frac{\pi r \cot \beta}{\sin \gamma}$$

$$m2 = mn - 2n$$

$$= Am \sin \gamma - r \operatorname{vers} A'O2'$$

Set a transit on lagging at crown of arch, over point of intersection of center line of arch and spiral to be located. Turn off from center line the angle γ. The direction of the straight line AB is thus fixed. Next lay off the distances Am and m2, giving the point 2 on the spiral. Several

Fig. 18.



points may in a similar way be determined, and the curve then drawn by applying to the known points a long, narrow, flexible straight-edge.

Other Systems for Cutting the Voussoirs. Various other systems for cutting the voussoirs of the helicoidal oblique arch are recognized, which the student will find explained in Dr Frank, Whimar, and other writers' works; but the system here presented is generally regarded as simplest, most economical of material, and as sufficiently exact in practical application.

Other Forms of Oblique Arches.

Besides the Helicoidal oblique arch, may be noted, among others:

False Skew Arch, built in the same manner as a right arch, the ring stones merely being cut to conform to the plane of the arch face. (See Fig. 19). Frequently adopted where the obliquity is not great, but has an element of weakness in the mode in which the thrust must be carried from crown to abutment on one side of the centre.

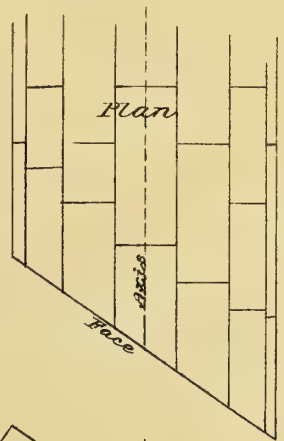


Fig. 19.

Ribbed Oblique Arch. (Fig. 20).

This consists of a series of independent, narrow, elliptical, right arches. The ribs present a series of projecting edges within the interior of the arch.

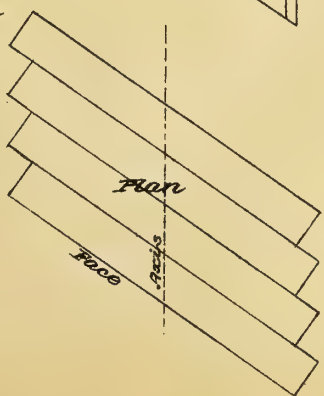


Fig. 20.

Convergent Arch.

In case of a long oblique arch, the central portion may be built as a right-arch, and toward the ends the coursing joints, which in the central portion are parallel to the axis, may be gradually curved around until normal to the faces, giving the so-called Convergent arch (see Fig. 21).

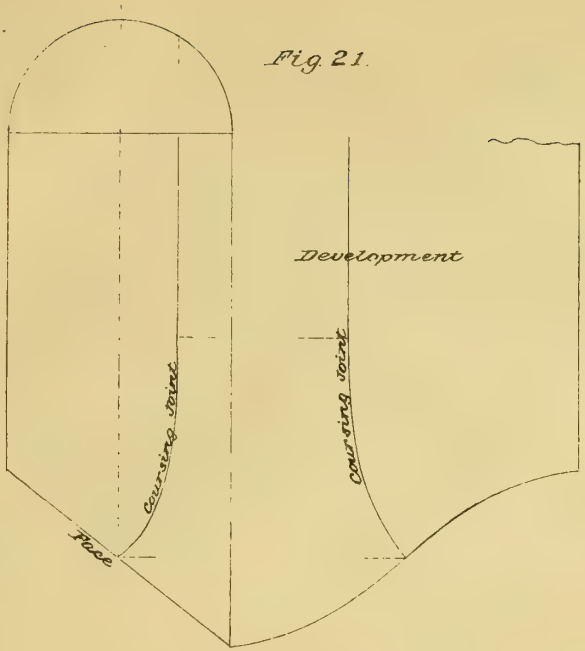


Fig. 21.

Logarithmic Oblique Arch. (Fig. 22).

So-called because Napierian logarithms enter into the equation of the joint curves. This arch has certain features of similarity to the helicoidal arch.

The heading joints are planes parallel to the arch faces, and therefore intersect intrados and extrados in elliptical curves. The coursing joints are warped surfaces normal to all planes parallel to the arch faces, and

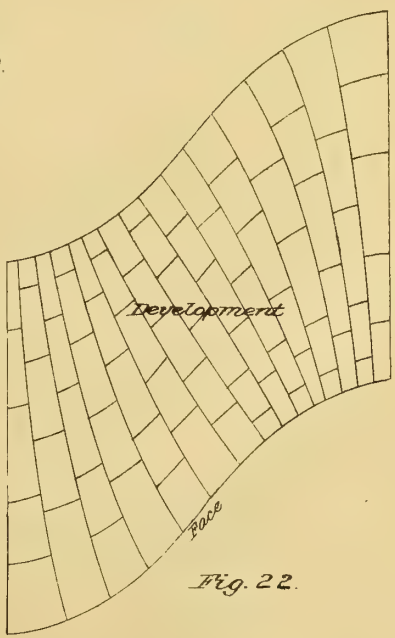


Fig. 22.

therefor normal to the heading joints. A coursing joint is generated by a normal line moving along the axis of the arch as one directrix, and along a curve (commonly taken in the intrados) normal to the arch face, as the other directrix.

According to the usual arrangment of voussoirs, each stone will have an exact duplicate in the corresponding voussoir on the opposite side of the arch, but in either half of the arch, as determined by the center line, no two stones will be alike. The difficulties of stone cutting are clearly much increased over those in the helicoidal method, and the advantages of the system are not such as to commend it to preference.

Cow's Horn Arch. In a few exceptional cases, this form of arch has been employed, its distinguishing feature being that the intrados is a warped surface of the variety known as the Cow's Horn.

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